数理解析研究所講究録
521巻 1984年 24-28

Nonstandard theory of functional and arithmetical divisors

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Let $\mathbb{Q}$ be the field of rational numbers and $^*\mathbb{Q}$ an ultrapower of $\mathbb{Q}$. Let Γ be an irreducible plane curve definede by

$$f(X,Y)=0$$

where f is an irreducible polynomial whose coefficients are contained in $\mathbb{Q}$.

Assume that there are infinitely many rational points on Γ, then there exists a nonstandard rational point (x,y) on Γ. Since $\mathbb{Q}$ is relatively algebraically closed in $^*\mathbb{Q}$, x is transcendental over $\mathbb{Q}$. Hence the field $F=\mathbb{Q}(x,y)$ is the function field of one variable of Γ. Now we have the situation

$$\begin{array}{c} ^*\mathbb{Q} \\ | \\ F=\mathbb{Q}(x,y) \\ | \\ \mathbb{Q} \end{array} \qquad (1)$$

In this situation we have two kinds of prime divisors, arithmetica prime divisors and functional prime divisors. By arithmetical prime divisors, we mean prime divisors of $^*\mathbf{Q}$, namely archimedean absolute value or nonarchimedean valuation. Nonarchimedean valuations are correspond to prime numbers of $^*\mathbf{Q}$. By functional prime divisors we mean nontrivial valuations of F over $\mathbf{Q}$.

In their paper, A. Robinson and P. Roquette gave a relation between arithmeticl and functional prime divsors.

Lemma 1. *Every functional prime divisor is induced by an arithmetical prime divisor.*

Using this lemma, they gave an another proof of so called the first fundamental inequality of Siegel;

Let u,v be any nonconstants of F. For any $\varepsilon>0$, there exists a constant C such that for any rational point P on Γ, if $H(P)>C$, then

$$\left|\frac{\log H(u(P))}{\log H(v(P))}-\frac{[F;\mathbf{Q}(u)]}{[F;\mathbf{Q}(v)]}\right|<\varepsilon$$

where H is the height function.

Our aim is to generalize the siegels inequality to be applicable to algebraic surfaces.

Theorem. *Let V be an irreducible algebraic surface defined over $\mathbf{Q}$ and G the function field of V over $\mathbf{Q}$. Let $t\in G$ be transcendental over $\mathbf{Q}$ and u,v transcendental over $\mathbf{Q}(t)$. For any $\varepsilon>0$, there exsists a constant C such that for any rational point P on V if*

$$\min(H(u(P)),H(v(P)))>H(t(P))^{C}>C^{C},$$

then

$$\left|\frac{\log H(u(P))}{\log H(v(P))}-\frac{[G;\mathbf{Q}(t,u)]}{[G;\mathbf{Q}(t,v)]}\right|<\varepsilon.$$

To prove Theorem, we intriduce H-convex subfields of $^*\mathbf{Q}$. A subfield K of $^*\mathbf{Q}$ is called H-convex if $H(a)\leqq H(b)$ and $b\in K$ imply $a\in K$.

Let Q_1 be a H-convex subfield of $^*\mathbf{Q}$ and F is a function field of one variable over Q_1, embedded into $^*\mathbf{Q}$. Now we have the same situation as before, but unfortunately Lemma 1 does not hold in this case. Hence we must slightly modify Lemma 1 as follows.

Let $R_\infty=\{\beta/\alpha\in{}^*\mathbf{Q}\mid |\beta/\alpha|<\gamma \text{ for some } \gamma\in Z_1\}$, then R_∞ is a valuation ring whose maximal ideal is $\{\beta/\alpha\in{}^*\mathbf{Q}\mid |\beta/\alpha|<1/\gamma \text{ for all } \gamma\in Z_1\}$

where $Z_1={}^*\mathbf{Z}\cap Q_1$.

Let $R=\{\beta/\alpha\in{}^*\mathbb{Q}\mid \alpha\in Z_1,\ \beta\in{}^*\mathbb{Z}\}$ and I a maximal ideal of R. Then the local ring of I is a valuation ring which we denote by R_I.

If $F\cap R_\infty$ is not trivial, namely $F\not\subset R_\infty$, then $F\cap R_\infty$ is a valuation ring. Since $F\cap R_\infty\supset Q_1$, this valuation ring yields a functional prime P of F. We say that P is induced by the archimedean prime. If $R_I\cap F$ is not trivial, then $R_I\cap F$ also yields a functional prime P of F. We say that P is induced by I.

Lemma 2. *Every functional prime P is induced by the archimedean prime or a maximal ideal of R.*

Proof. By the theorem of Riemann-Roch, there exists β/α which admits P as its only pole. If $|\beta/\alpha|>\gamma$for all $\gamma\in Z_1$, then $\beta/\alpha\notin R_\infty$. Hence $\beta/\alpha\notin R_\infty\cap F$. Then the functional prime induced by the archimedean prime is a pole of β/α. Since P is the only functional prime which is a pole of β/α, P is induced by the archimedean prime. If $|\beta/\alpha|<\gamma$ for some $\gamma\in Z_1$, then $\alpha\in{}^*\mathbf{Z}-Z_1$ because β/α is a nonconstant. Let I_α be a maximal ideal of which cotains α. Then the local ring of I_α does not contain β/α.

Then $\beta/\alpha \notin R_{I_\alpha} \cap F$. By the same arguments as above, P is induced by I_α. Lemma 2 is proved.

For details of proof of Theorem, Please refer to [2].

Refernces

[1] A. Robinson and P. Roquette, On the finiteness theorem of Siegel and Mahler concerning diophantine equations, J.Number Theory 7 (1975), 121-176.

[2] M. Yasumoto, Nonstandard arithmetic of function fields over H-convex subfields of *Q, Journal fur die reine und angewandte Mathmatik, 342 (1983), 1-11.